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NASH SOLUTION BY EXTREMUM SEEKING CONTROL APPROACH

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Nash Solution by Extremum Seeking Control Approach[1]

Yaodong Pan[2] Tankut Acarman[3] Ümit Özgüner[4]

Abstract

In this paper, we propose an algorithm to solve the Nash equilibrium solution for an n-person noncooperative dynamic game by the extremum seeking control approach with sliding mode. For each player, a switching function is defined as the difference between the player's cost function and a reference signal. The extremum seeking controller for each player is designed so that the system converges to a sliding boundary layer defined in the vicinity of a sliding mode corresponding to the switching function and inside the boundary layer, the cost function tracks the reference signal and converges to the Nash equilibrium solution.

Keyword: Noncooperative Game, Nash Equilibrium Solution, Extremum Seeking, Sliding Model

1 Introduction

For an n-person noncooperative dynamic game, each player defines a cost function and adjusts some of the control parameters to minimize his own cost function [1][2] to find a Nash equilibrium solution. When the cost function as a measurable variable or a combination of some measurable variables can not be exactly formulated, i.e. when the form of the cost function is not given mathematically although it is measurable, extremum seeking control with sliding mode can be used to solve for the Nash solution.

Extremum seeking control approaches have been proposed to find a setpoint and/or track a varying setpoint so that a cost function (which may be unknown) of the system reaches the extremum[3][4][5][6]. The extremum seeking controllers with sliding mode have been proposed[7][8][9][10], and can be explained by the configuration in Figure 1. The switching function $s(t)$ is defined as

$$s(t) = y(t) - g(t)$$

where $g(t)$ is a reference signal. The setpoint for the minimum (or maximum) can be reached no matter how the plant changes. With this control method, the sliding mode $s(t) = 0$ happens, the system oscillates in the vicinity of the sliding mode $s(t) = 0$, i.e. oscillates inside a sliding boundary layer $|s(t)| \leq \epsilon$, and a minimum or maximum point can be reached in the sliding mode, as shown in Figure 1. Designing an extremum

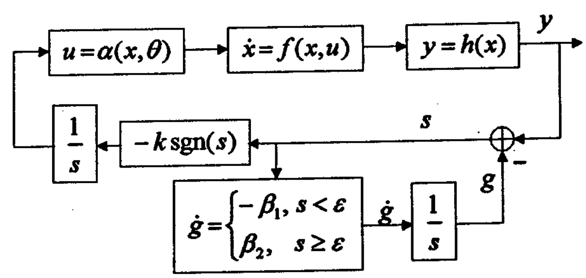

Figure 1: Extremum Seeking Control Using Sliding Mode

seeking controller for each player ensures that the dynamic game system converges to the Nash equilibrium point.

The arrangement of the paper is as follows. Section 2 describes the problem formulation; Section 3 proposes the extremum seeking algorithm using sliding mode to calculate the Nash equilibrium solution; and Section 4 gives simulation results.

2 Problem Formulation

Consider an n-person noncooperative dynamic game described by a nonlinear system

$$\frac{d}{dt}x(t) = f(x(t), u_1(t), u_2(t), \cdots, u_n(t)) \qquad (1)$$

with cost function for i-th player

$$J_i(t) = J_i(x(t)), \quad (i \in N) \qquad (2)$$

where N is the index set of player defined as

$$N = \{1, 2, \cdots, n\},$$

$x(t) \in R^m$, $u_i(t) \in R$ $(i \in N)$, and $J_i(t) \in R$ $(i \in N)$ are the state variable, the i-th player's control input,

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and the i-th player's cost function, respectively. The functions, $f(x(t), u_1(t), u_2(t), \cdots, u_n(t))$ and $J_i(x)$ ($i \in N$) are assumed to be smooth.

Assumption 1 *There exist smooth control laws*

$$u_i(t) = \alpha_i(x(t), \theta_i), \quad (i \in N) \tag{3}$$

for all players to stabilize the above nonlinear system (1), where $\theta_i \in \Theta_i$ ($i \in N$) is a control parameter.

With the control input (3), the closed-loop system of the nonlinear system (1) is determined by

$$\frac{d}{dt}x(t) = f(x(t), \alpha_1(x(t), \theta_1), \alpha_2(x(t), \theta_2), \cdots, \alpha_n(x(t), \theta_n))$$

Assumption 2 *There exist a smooth function x_e: $R \to R^n$ such that*

$$f(x(t), \alpha_1(x(t), \theta_1), \alpha_2(x(t), \theta_2), \cdots, \alpha_n(x(t), \theta_n)) = 0$$
$$\Updownarrow$$
$$x = x_e(\theta_1, \theta_2, \cdots, \theta_n)$$

i.e., every n-tuple of the control parameters $\theta_i \in \Theta_i$ ($i \in N$) determines a unique equilibrium point $x_e(\theta_1, \theta_2, \cdots, \theta_n)$.

Assumption 3 *The static performance map at the equilibrium point $x_e(\theta_1, \theta_2, \cdots, \theta_n)$ from a n-tuple of $\theta_i \in \Theta_i$ ($i \in N$) to $J_i(t)$ represented by*

$$\begin{aligned} J_i^e &= J(x_e(\theta_1, \theta_2, \cdots, \theta_n)) \\ &= J(\theta_1, \theta_2, \cdots, \theta_n), \quad , (i \in N) \end{aligned}$$

is smooth and has a unique Nash equilibrium solution

$$J^*(\theta_1^*, \theta_2^*, \cdots, \theta_n^*)$$

at point (θ_1^*, θ_2^*, $\cdots$, θ_n^*) such that

$J_i(\theta_1, \cdots \theta_n)$

$$\begin{aligned} J_i^*(\theta_1^*, \theta_2^*, \cdots, \theta_i^*, \cdots, \theta_n^*) &\leq \cancel{J(\theta_1^*, \theta_2^*, \cdots, \theta_i, \cdots, \theta_n^*)} \\ \forall \theta_i &\in \Theta_i, (i \in N) \end{aligned}$$

Assumption 4 *The partial derivative of the static performance map J_i^e ($i \in N$) satisfies*

unclear why > not sufficient.

$$|\frac{\partial}{\partial \theta_i} J_i^e| >> |\frac{\partial}{\partial \theta_j} J_i^e|, \quad \forall j \neq i \quad (i \in N)$$

The control objective is to solve the Nash equilibrium solution $J^*(\theta_1^*, \theta_2^*, \cdots, \theta_n^*)$ by adjusting the parameters θ_i by each player ($i \in N$) separately.

3 Extremum Seeking with Sliding Mode

To design an extremum seeking controller with sliding mode for the i-th player ($i \in N$), a switching function is defined as

$$s_i(t) = J_i(t) - g_i(t) \tag{4}$$

where the reference signal $g_i(t) \in R$ is determined by

$$\dot{g}_i(t) = \beta_i(t), \tag{5}$$

where the time-varying parameter $\beta_i(t)$ will be given later. Then a sliding boundary layer based on the above switch function is defined as

$$|s_i(t)| \leq \epsilon_i \tag{6}$$

where $\epsilon_i > 0$ is a small positive constant.

Let the variable structure control law be

$$v_i(t) = -k_i \text{sgn}(s_i(t)) \tag{7}$$

and the parameter θ_i satisfy

$$\dot{\theta}_i(t) = v_i(t)$$

where k_i is a small enough positive constant.

Assumption 5 *The dynamic system given in (1) is much faster than the one of the parameter θ_i's adjusting process, i.e.*

$$|\frac{d}{dt}x(t)| >> |\frac{d}{dt}\theta_i|.$$

Therefore in the design of the extremum seeking controller, the cost function $J_i(t)$ can be replaced by the static performance map

$$J_i^e = J_i(\theta_1, \theta_2, \cdots, \theta_n).$$

Assumption 5 is reasonable once k_i is small enough.

Assumption 6 *The setpoint (θ_1^*, θ_2^*, $\cdots$, θ_n^*) corresponding to the Nash equilibrium solution is in the vicinity of the initial n-tuple of $\theta_i(0)$ ($i \in N$). Thus the partial derivative of the cost function $J_i(t)$ on θ_i is bounded by a positive constant γ_i, i.e.,*

$$|\frac{\partial}{\partial \theta_i} J_i(\theta_1, \theta_2, \cdots, \theta_i, \cdots, \theta_n)| \leq \gamma_i \tag{8}$$

Based on the above assumptions, the derivative of the switching function $s_i(t)$ is given by

$$\begin{aligned} \frac{d}{dt}s_i(t) &= \sum_{j=1}^{n} \frac{\partial}{\partial \theta_j} J_i(\theta_1, \theta_2, \cdots, \theta_n)\dot{\theta}_j(t) - \dot{g}_i(t) \\ &= -W_i(\theta_1, \theta_2, \cdots, \theta_n)k_i \text{sgn}(s_i(t)) - \beta_i(t) \end{aligned}$$

where $W(\theta_1, \theta_2, \cdots, \theta_n)$ is determined by

$$W_i(\theta_1, \theta_2, \cdots, \theta_n) = \sum_{j=1}^{N} \frac{\partial}{\partial \theta_j} J_i(\theta_1, \theta_2, \cdots, \theta_n) \frac{k_j \text{sgn}(s_j(t))}{k_i \text{sgn}(s_i(t))}$$

According to Assumptions 4 and 6 and if the positive constant k_i ($i \in N$) is bounded by some constant, it can be shown that $W_i(\theta_1, \theta_2, \cdots, \theta_n)$ is bounded. To simplify the notations, it is assumed to be bounded by γ_i ($i \in N$), i.e.

$$|W_i(\theta_1, \theta_2, \cdots, \theta_n)| \leq \gamma_i. \quad (i \in N)$$

Define a Lypunov function as

$$V_i(t) = \frac{1}{2} s_i^2(t) \tag{9}$$

Then

$$\begin{aligned} \frac{d}{dt} V_i(t) &= s_i(t) \frac{d}{dt} s_i(t) \\ &= -W_i(\theta_1, \theta_2, \cdots, \theta_n) k_i |s_i(t)| - s_i(t) \beta_i(t) \end{aligned}$$

According to sliding mode control theory, to ensure the convergence to the sliding mode $s_i(t) = 0$ or the sliding boundary layer $|s_i(t)| \leq \epsilon_i$, the above derivative must be negative. Therefore the time-varying parameter $\beta_i(t)$ outside the sliding boundary layer (6) is chosen as

$$\beta_i(t) = \begin{cases} -\hat{\beta}_i, & s_i(t) < -\epsilon_i \\ \bar{\beta}_i, & s_i(t) > \epsilon_i \end{cases} \tag{10}$$

where $\hat{\beta}_i$ and $\bar{\beta}_i$ are positive constants satisfying

$$\begin{aligned} \hat{\beta}_i &> \gamma_i k_i + \sigma_i \\ \bar{\beta}_i &> \gamma_i k_i + \sigma_i \end{aligned}$$

σ_i is a positive constant. Thus the following holds.

$$\frac{d}{dt} V_i(t) \leq -\sigma_i |s_i(t)|, \quad |s_i(t)| > \epsilon_i \tag{11}$$

which means that the system will enter the sliding boundary layer $|s_i(t)| \leq \epsilon_i$ in a finite time and stay there ~~since then~~. after that.

Inside the sliding boundary layer $|s_i(t)| \leq \epsilon_i$, the time-varying parameter $\beta_i(t)$ is chosen as

$$\beta_i(t) = \begin{cases} -\tilde{\beta}_i, & -\epsilon_i \leq s_i(t) < \epsilon_i \\ 2\epsilon_i \delta(t - t_0), & s_i(t) = \epsilon_i \end{cases} \tag{12}$$

where $\tilde{\beta}_i$ is a positive constant satisfying

$$\tilde{\beta}_i > \gamma_i k_i + \sigma_i,$$

$\delta(t - t_0)$ is the impulse function defined as

$$\begin{aligned} \delta(t - t_0) &= \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases} \\ \int_{t_0}^{t_0^+} \delta(t - t_0) dt &= 1 \end{aligned}$$

t_0 is the time instant when $s_i(t)|_{t=t_0} = \epsilon_i$.

With the parameter $\beta_i(t)$ designed above, inside the sliding boundary $|s_i(t)| \leq \epsilon_i$ except for one of the boundaries, $s_i(t) = \epsilon_i$, the reference signal $g_i(t)$ keeps decreasing with

$$\dot{g}_i(t) = -\tilde{\beta}_i.$$

At the same time, the cost function $J_i(t)$ may increase or decrease but the absolute value of the change rate of $J_i(t)$ is less than the one of $g_i(t)$ as

$$\begin{aligned} |\dot{J}_i(t)| &= |W_i(\theta_1, \theta_2, \cdots, \theta_n)| k_i \\ &\leq \gamma_i k_i < \tilde{\beta}_i - \sigma_i < \tilde{\beta}_i = |\dot{g}_i(t)| \end{aligned}$$

Therefore $s_i(t) = J_i(t) - g_i(t)$ will increase, i.e., the system will move toward the sliding boundary $s_i(t) = \epsilon_i$ and reach the boundary at some time instant t_0. Then with the adjusting rule $\dot{g}_i(t) = -2\epsilon_i \delta(t - t_0)$, the system will move to the another boundary $s_i(t) = -\epsilon_i$ as

$$\begin{aligned} s_i(t_0) &= J_i(t_0) - g_i(t_0) = \epsilon_i \\ s_i(t_0^+) &= J_i(t_0^+) - g_i(t_0^+) \\ &= (J_i(t_0) - g_i(t_0)) - 2\epsilon_i \int_{t_0}^{t_0^+} \delta(t - t_0) dt \\ &= -\epsilon_i \end{aligned}$$

After then, the system will move from the boundary $s_i(t) = -\epsilon_i$ to another one $s_i(t) = \epsilon_i$, again while the reference signal $g_i(t)$ keeps decreasing. In this way, the system will vibrate inside the sliding ~~boundary layer~~ $|s_i(t)| \leq \epsilon_i$.

Continued improve Nash

It is assumed that the system reaches the boundary $s_i(t) = \epsilon_i$ at time instants $t = t_0, t_1, t_2, \cdots$ as shown in Figure 2. If the sliding boundary layer is chosen to

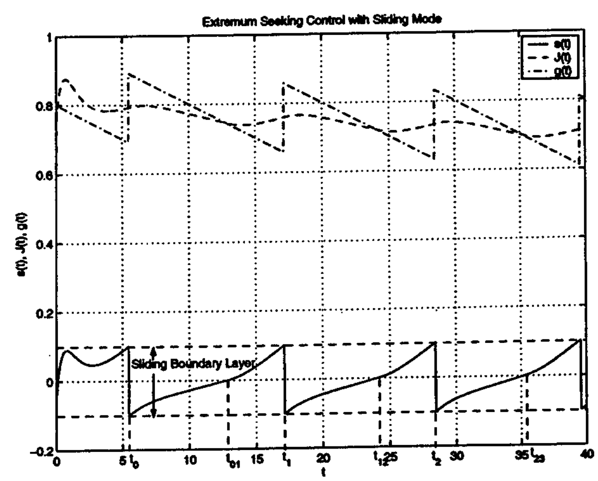

Figure 2: Extremum Seeking Control with Sliding Mode

be narrow enough, then it is reasonable to assume that every period $[t_i, t_{i+1})$ ($i = 0, 1, 2, \cdots$) is very short so that the function $W_i(\theta_1(t), \theta_2(t), \cdots, \theta_n(t))$, denoted as

$$\alpha_{i,j} = W_i(\theta_1(t), \theta_2(t), \cdots, \theta_n(t))|_{t \in [t_j, t_{j+1})}, (j = 0, 1, 2, \cdots)$$

is a constant in each period and satisfies

$$|\alpha_{i,j}| \le \gamma_i. \quad (j = 0, 1, 2, \cdots)$$

Now let's show that the cost function $J_i(t)$ will decrease while the system vibrates inside the sliding boundary layer.

At the time instant t_0,

$$s_i(t_0) = J_i(t_0) - g_i(t_0) = \epsilon_i.$$

Then at the time instant t_0^+,

$$\begin{aligned} J_i(t_0^+) &= J_i(t_0) \\ g_i(t_0^+) &= g_i(t_0) + 2\epsilon_i, \\ s_i(t_0^+) &= -\epsilon_i. \end{aligned}$$

Let t_{01} denote the time instant when $s_i(t)|_{t=t_{01}} = 0$ $(t_0 < t_{01} < t_1)$. For $t_0^+ \le t < t_{01}$, the followings hold.

$$\begin{aligned} s_i(t) &= J_i(t) - g_i(t) < 0, \\ \dot{J}_i(t) &= -\alpha_{i,0} k_i \mathrm{sgn}(s_i(t)) = \alpha_{i,0} k_i, \\ \dot{g}_i(t) &= -\tilde{\beta}_i < 0 \end{aligned}$$

which yield

$$\begin{aligned} J_i(t) &= J_i(t_0) + \alpha_{i,0} k_i (t - t_0) \\ g_i(t) &= g_i(t_0) + 2\epsilon_i - \tilde{\beta}_i (t - t_0). \end{aligned}$$

According to

$$s_i(t_{01}) = J_i(t_{01}) - g_i(t_{01}) = 0$$

the time instant t_{01} can be found as

$$t_{01} = t_0 + \frac{\epsilon}{\tilde{\beta}_i + \alpha_{i,0} k_i}$$

and $J_i(t)$ and $g_i(t)$ at $t = t_{01}$ are given by

$$\begin{aligned} J_i(t_{01}) &= J_i(t_0) + \frac{\epsilon_i \alpha_{i,0} k_i}{\tilde{\beta}_i + \alpha_{i,0} k_i} \\ g_i(t_{01}) &= g_i(t_0) + 2\epsilon_i - \frac{\epsilon_i \tilde{\beta}_i}{\tilde{\beta}_i + \alpha_{i,0} k_i} \end{aligned}$$

For $t_{01} < t < t_0$, the followings hold.

$$\begin{aligned} s_i(t) &= J_i(t) - g_i(t) > 0, \\ \dot{J}(t) &= -\alpha_{i,0} k_i, \\ \dot{g}_i(t) &= -\hat{\beta}_i < 0 \end{aligned}$$

which yield

$$\begin{aligned} J_i(t) &= J_i(t_{01}) - \alpha_{i,0} k_i (t - t_{01}) \\ g_i(t) &= g_i(t_{01}) - \tilde{\beta}_i (t - t_{01}) \end{aligned}$$

According to

$$s_i(t_1) = J_i(t_1) - g_i(t_1) = \epsilon_i$$

the time instant t_1 can be found as

$$t_1 = t_{01} + \frac{\epsilon}{\tilde{\beta}_i - \alpha_{i,0} k_i},$$

and $J_i(t)$ and $g_i(t)$ at $t = t_1$ are determined by

$$\begin{aligned} J_i(t_1) &= J_i(t_0) - 2\epsilon_i \frac{\alpha_{i,0}^2 k_i^2}{\tilde{\beta}_i^2 - \alpha_{i,0}^2 k_i^2} \\ g_i(t_1) &= g_i(t_0) - 2\epsilon_i \frac{\alpha_{i,0}^2 k_i^2}{\tilde{\beta}_i^2 - \alpha_{i,0}^2 k_i^2} \end{aligned}$$

i.e., the cost function $J_i(t)$ and the reference signal $g_i(t)$ decrease in the period $[t_0, t_1)$ as

$$\begin{aligned} J_i(t_1) - J_i(t_0) &= -2\epsilon_i \frac{\alpha_{i,0}^2 k_i^2}{\tilde{\beta}_i^2 - \alpha_{i,0}^2 k_i^2} < 0 \\ g_i(t_1) - g_i(t_0) &= -2\epsilon_i \frac{\alpha_{i,0}^2 k_i^2}{\tilde{\beta}_i^2 - \alpha_{i,0}^2 k_i^2} < 0. \end{aligned}$$

In a similar way, it can be shown that before the Nash equilibrium solution is reached, the followings hold.

$$\begin{aligned} J_i(t_0) &> J_i(t_1) > J_i(t_2) > \cdots = 0 \\ g_i(t_0) &> g_i(t_1) > g_i(t_2) > \cdots = 0 \end{aligned}$$

When the Nash Solution is reached at a time instant t_m, i.e. when $\alpha_{i,j} = 0$ $(j = m, m+1, m+2, \cdots)$, $J_i(t)$ and $g_i(t)$ will keep to be a constant.

Theorem 1 *Consider the dynamic noncooperative game described by the state equation in (1) with the control input in (3), the sliding mode controller with extremum seeking control approach for the i-th player $(i \in N)$ designed as*

$$\begin{aligned} s_i(t) &= J_i(t) - g_i(t) \\ \dot{\theta}_i = &= v_i(t) \\ v_i(t) &= -k_i \mathrm{sgn}(s_i) \\ \dot{g}_i(t) &= \begin{cases} -\hat{\beta}_i, & s_i(t) < -\epsilon_i \\ -\tilde{\beta}_i, & -\epsilon_i \le s_i(t) < \epsilon_i \\ 2\epsilon_i \delta(t - t_0), & s_i(t) = \epsilon_i \\ \bar{\beta}_i, & s_i(t) > \epsilon_i \end{cases} \end{aligned}$$

ensures that the cost functions $J_i(t)$ $(i \in N)$ are minimized to get the Nash equilibrium solution $J^*(\theta_1^*, \theta_2^*, \cdots, \theta_n^*)$.

Remark 1 *The variable structure control rule for the i-th player $(i \in N)$ may be replaced by*

$$v_i(t) = -k_i \mathrm{sgn}(sin(\omega_i s_i(t) \pi / 2 / \epsilon_i)), \quad (i \in N) \quad (13)$$

where $\omega_i \ge 1$ is a positive number, then the amplitude of the vibration on the cost function $J_i(t)$ becomes smaller for larger ω_i , which still results in a stable extremum seeking control.

Remark 2 *To implement the proposed algorithm for the sampled-data system and also to simplify the controller, the reference signal $g_i(t)$ $(i \in N)$ can be modified as*

$$\dot{g}_i(t) = \begin{cases} -\hat{\beta}_i, & s_i(t) < \epsilon_i \\ \bar{\beta}_i, & s_i(t) \geq \epsilon_i \end{cases} (i \in N) \qquad (14)$$

where $\bar{\beta}_i$ satisfies

$$l_i \bar{\beta}_i T_i = 2\epsilon_i$$

T_i is the sampling interval and l_i is a positive constant which indicates the number of the sampling intervals when $\dot{g}_i(t) = \bar{\beta}_i$ $(i \in N)$.

4 Examples

Consider a two-person noncooperative dynamic game described by a second-order linear system with unknown parameters.

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 0.2 \\ 0.3 & -1 \end{bmatrix} x(t) \\ &+ \begin{bmatrix} 0.5(u_1(t) - 2 - 0.1u_2(t))^2 + 1.0 \\ 0.7(u_2(t) - 1 - 0.2u_1(t))^2 + 0.5 \end{bmatrix}. \end{aligned}$$

The cost functions for two players are respectively defined as

$$\begin{aligned} J_1(t) &= x_1(t) \\ J_2(t) &= x_2(t) \end{aligned}$$

The control input is chosen to be the control parameter, i.e.

$$u_i(t) = \theta_i(t), \quad (i = 1, 2)$$

Then it is clear that the Nash equilibrium point is given by

$$\begin{aligned} \theta_1^* &= 1.4286 \\ \theta_2^* &= 2.1429. \end{aligned}$$

The proposed extremum seeking control algorithm is implemented for the above system with sampling interval as $T = 0.01$ second and other parameters as

$$\begin{aligned} \bar{\beta}_i &= 0.005, \quad \hat{\beta}_i = 5.0, \quad l_i = 2 \\ \epsilon_i &= 0.05, \quad k_i = 0.01, \quad (i = 1, 2) \end{aligned}$$

The simulation results are given in Figure 3, which shows that the system enters the sliding boundary layer in a finite time and then oscillates inside the layer while the cost function keeps decreasing with oscillation until the Nash equilibrium point is reached. The amplitude of the vibration can be reduced by choosing a smaller boundary layer ϵ. Figure 4 are simulation results with

$$\epsilon_i = 0.01, \quad \hat{\beta}_i = 1.(i = 1, 2)$$

Using the control laws given in Remark 1, as shown in Figure 5, results in higher control accuracy with a larger constants ω_i $(i = 1, 2)$.

5 Conclusion

The extremum seeking control approach with sliding mode proposed in [7][8][9][10] was implemented in an n-person noncooperative dynamic game to calculate the Nash equilibrium solution. With the designed controller for each player in the game, the system enters a sliding boundary layer and stays there while the cost function decreases with oscillating behavior, until the Nash equilibrium solution is reached. The simulation result show the effectiveness.

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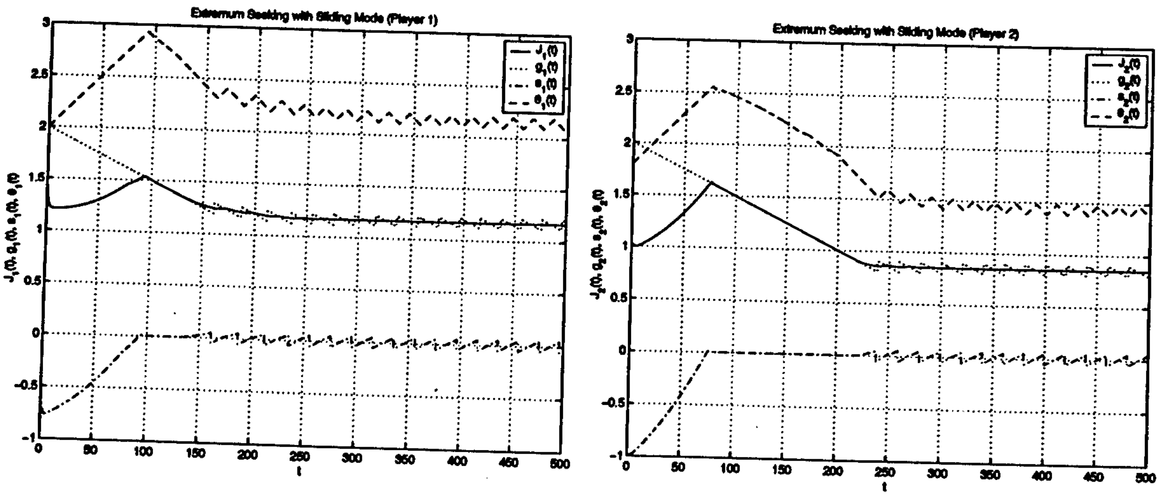

Figure 3: Nash Solution by Extremum Seeking Control($\epsilon_1 = \epsilon_2 = 0.05$)

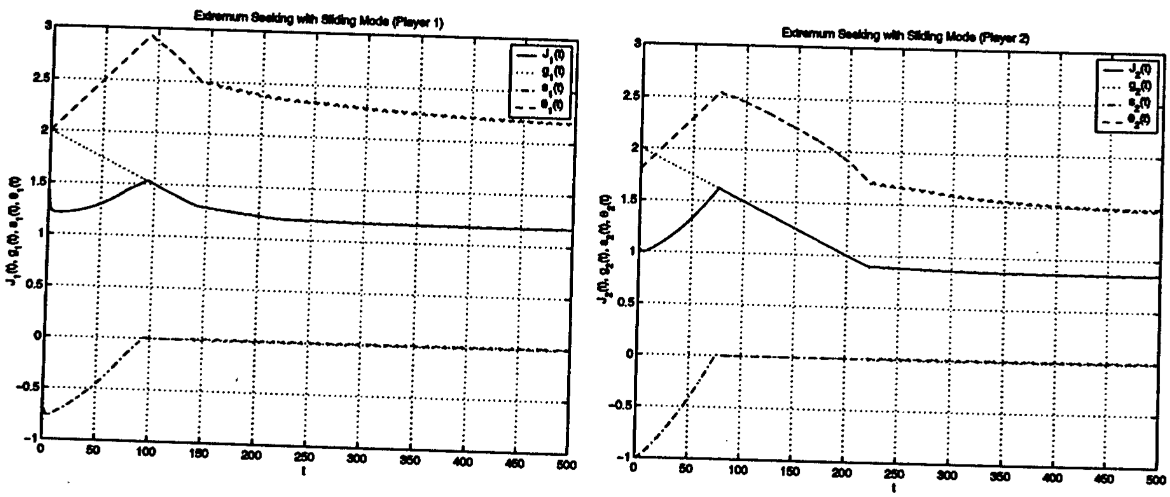

Figure 4: Nash Solution by Extremum Seeking Control($\epsilon_1 = \epsilon_2 = 0.01$)

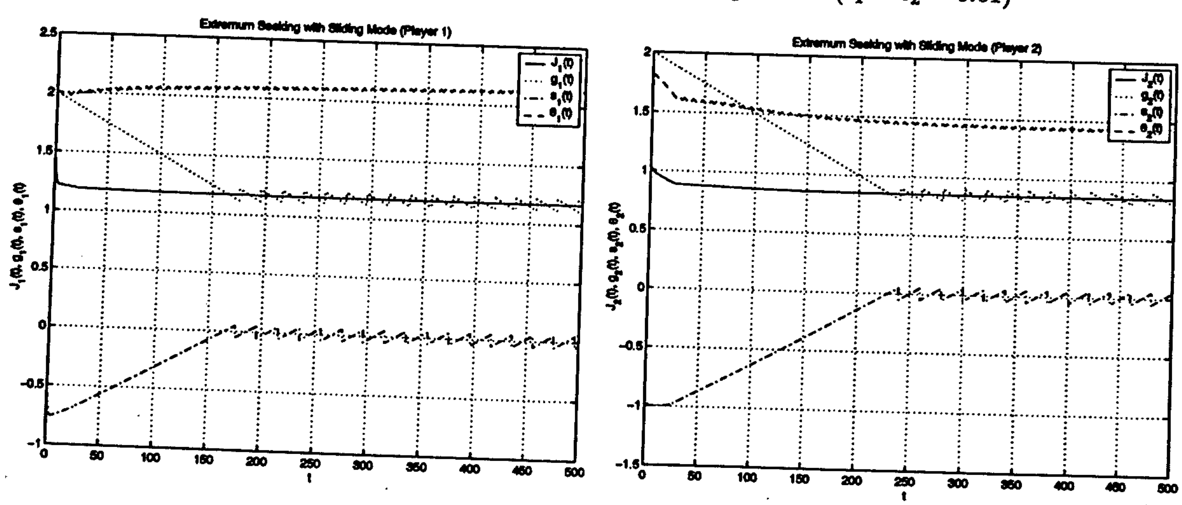

Figure 5: Nash Solution by Extremum Seeking Control($\epsilon_1 = \epsilon_2 = 0.05$, $\omega_1 = \omega_2 = 10$)